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Very Differentiable and Generic Frechet Differentiable Convex Functions on Banach Spaces

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Using the extension of convex functions on a Banach space X to the bidual space X^{**}, we introduce and study a kind of differentiability of convex functions. Moreover, we discuss Asplund spaces. © 1999 Academic Press

1. INTRODUCTION

Throughout this paper, X always denotes a Banach space. X^*, X^{**}, and X^{***} stand for the dual space, bidual space, and tridual space of X, respectively. We always regard X as a subspace of X^{**}. For $x \in X$, $B_X(x, r)$, and $B_{X^{**}}(x, r)$ stand for the ball with center x and radius r in X and X^{**}, respectively. For a subset A of X, we denote the weak* closure of A in X^{**} by $w^*\text{-cl}_{X^{**}}(A)$. w and w^* always stand for the weak topology and the weak* topology, respectively.

Let D be an open convex subset of X, $f\colon D \to R$ a continuous convex function. For $x \in D$, $\partial f(x)$ denotes the subdifferential of f at x, that is,

$$\partial f(x) = \{x^* \in X^*; \langle x^*, y - x\rangle \le f(y) - f(x), y \in D\}.$$

It is well known that ∂f is a $\|\cdot\|$-w^* upper semicontinuous set-valued mapping and $\partial f(x)$ is a nonempty w^* compact convex subset of X^* for all $x \in D$. We say that f is Gateaux differentiable at $x \in D$ if the limit

$$\lim_{t \to 0} \frac{f(x + th) - f(x)}{t}$$

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exists for all $h \in X$. This is equivalent to $\partial f(x)$ being a singleton. We say that f is Frechet differentiable at $x \in D$ if the limit

$$\lim_{t \to 0} \frac{f(x+th) - f(x)}{t}$$

exists uniformly with respect to h on the unit ball of X. It is known that f is Frechet differentiable at $x \in D$ iff $\partial f(x)$ is a singleton and ∂f is $\|\cdot\|$-$\|\cdot\|$ upper semicontinuous at x.

We say that X is an Asplund space if each continuous convex function defined on an open convex subset D of X is Frechet differentiable at each point in a dense G_δ subset of D. It is clear that X is an Asplund space iff for any open convex subset D of X and any continuous convex function f defined on D, the set $E = \{x \in D; \partial f(x)$ is a singleton and ∂f is $\|\cdot\|$-$\|\cdot\|$ upper semicontinuous at $x\}$ is a residual subset of D (i.e., E includes a dense G_δ subset of D). Asplund spaces are a very important subject in Banach spaces theory. A series of profound results on Asplund spaces were obtained (see [1, 6–12] and references therein).

In Section 2, we present some results about an extension of a continuous convex function defined on an open subset of a Banach space X to X^{**}. In Section 3, using the extension of convex functions, we introduce and study very differentiability of convex functions, which is a generalization of differentiability (or smoothness) of norms and, in general, is not any kind of β-differentiability. In Section 4, using the result that a continuous convex function f defined on an open convex subset D of a Banach space is Frechet differentiable at each point in a dense G_δ subset of D iff $D_f = \{x \in D; \partial f(x)$ is a weak compact set and ∂f is $\|\cdot\|$-w upper semicontinuous at $x\}$ is a residual subset of D, we give a characterization of Asplund spaces.

2. PRELIMINARIES

Let D be an open convex subset of X, f a continuous convex function defined on D. Pick a $x_0 \in D$ and $x_0^* \in \partial f(x_0)$; then $f(x) \geq f(x_0) + \langle x_0^*, x - x_0 \rangle$ for all $x \in D$. This implies that for each $x^{**} \in w^*\text{-cl}_{X^{**}}(D)$

$$\liminf_{x_D \xrightarrow{w^*} x^{**}} f(x) \geq f(x_0) + \langle x_0^*, x^{**} - x_0 \rangle > -\infty.$$

Hence we can define $\hat{f}\colon X^{**} \to R \cup \{+\infty\}$ such that

$$\hat{f}(x^{**}) = \begin{cases} \liminf\limits_{x_D \xrightarrow{w^*} x^{**}} f(x), & x^{**} \in w^*\text{-cl}_{X^{**}}(D), \\ +\infty, & \text{otherwise}. \end{cases}$$

This means that the following (I) and (II) hold:

(I) If a net $\{x_\alpha\}$ in D converges to some $x^{**} \in w^*\text{-cl}_{X^{**}}(D)$ with respect to the weak* topology of X^{**},

$$\liminf_{\alpha} f(x_\alpha) \geq \hat{f}(x^{**})$$

(II) For each $x^{**} \in w^*\text{-cl}_{X^{**}}(D)$ there is a net $\{x_\alpha\}$ in D such that $x_\alpha \xrightarrow{w^*} x^{**}$ and

$$\hat{f}(x^{**}) = \lim_{\alpha} f(x_\alpha).$$

PROPOSITION 2.1. *Let f be a continuous convex function defined on an open convex subset D of X. Then* $\text{epi}(\hat{f}) = w^*\text{-cl}_{X^{**}\times R}(\text{epi}(f))$. *Hence $\hat{f}$ is a proper weak* lower semicontinuous convex function on X^{**}.*

Proof. For any $(x^{**}, r) \in \text{epi}(\hat{f})$, $x^{**} \in w^*\text{-cl}_{X^{**}}(D)$ and there is $t \geq 0$ such that $r = \hat{f}(x^{**}) + t$. Pick a net $\{x_\alpha\}$ in D such that $x_\alpha \xrightarrow{w^*} x^{**}$ and $f(x_\alpha) \to \hat{f}(x^{**})$. It follows that $(x_\alpha, f(x_\alpha) + t) \xrightarrow{w^*} (x^{**}, r)$ in $X^{**} \times R$. Hence $(x^{**}, r) \in w^*\text{-cl}_{X^{**}\times R}(\text{epi}(f))$. On the other hand, for any $(x^{**}, r) \in w^*\text{-cl}_{X^{**}\times R}(\text{epi}(f))$, there is a net $\{x_\alpha\}$ in D and a net $\{t_\alpha\}$ in R^+ such that $(x_\alpha, f(x_\alpha) + t_\alpha) \xrightarrow{w^*} (x^{**}, r)$. Hence

$$r = \lim_{\alpha} (f(x_\alpha) + t_\alpha) \geq \liminf_{\alpha} f(x_\alpha) \geq \hat{f}(x^{**}).$$

This implies $(x^{**}, r) \in \text{epi}(\hat{f})$. ∎

PROPOSITION 2.2. *For each continuous convex function f defined on an open convex subset D of X, $\hat{f}$ is an extension of f and* $D \subset \text{int}_{X^{**}}(\text{dom}(\hat{f}))$, *where* $\text{dom}(\hat{f}) = \{x^{**} \in X^{**};\, \hat{f}(x^{**}) \in R\}$.

Proof. For each $x \in D$, by (I), $\hat{f}(x) \leq f(x)$. By (II) there is a net $\{x_\alpha\}$ in D such that $x_\alpha \xrightarrow{w} x$ (i.e., $\{x_\alpha\}$ converges to x with respect to the weak* topology of X^{**}) and $\hat{f}(x) = \lim_\alpha f(x_\alpha)$. Since a continuous convex function is lower semicontinuous with respect to the weak topology, $f(x) \leq \lim_\alpha f(x_\alpha) = \hat{f}(x)$. Hence $f(x) = \hat{f}(x)$ for all $x \in D$. By the continuity of f, for each $x \in D$ there are $\delta > 0$ and $M \in R$ such that $B_X(x, \delta) \subset D$ and $f(x) \leq M$ for all $x \in B_X(x, \delta)$. By the Goldstine theorem, $B_X(x, \delta)$ is weak* dense in $B_{X^{**}}(x, \delta)$. Hence, for each $x^{**} \in B_{X^{**}}(x, \delta)$, there is a net $\{z_\alpha\}$ in $B_X(x, \delta)$ such that $z_\alpha \xrightarrow{w^*} x^{**}$. It follows from (I) that $\hat{f}(x^{**}) \leq M$. Hence $x^{**} \in \mathrm{dom}(\hat{f})$. This shows that $D \subset \mathrm{int}_{X^{**}}(\mathrm{dom}(\hat{f}))$. ∎

LEMMA 2.1. *Let K be a nonempty bounded subset of X^* with $0 \in K$, and let $p_K(x) = \sup\{\langle x^*, x\rangle;\ x^* \in K\}$ for all $x \in X$ and $q_K(x^{**}) = \sup\{\langle x^{**}, x^*\rangle;\ x^* \in w^*\text{-cl}(\mathrm{co}(K))\}$ for all $x^{**} \in X^{**}$. Then $A = \{x^{**} \in X^{**};\ q_K(x^{**}) \leq 1\}$ is the weak* closure of $B = \{x \in X;\ p_K(x) \leq 1\}$.*

Proof. It is easy to verify that $p_K(x) = \sup\{\langle x^*, x\rangle;\ x^* \in w^*\text{-cl}(\mathrm{co}(K))\}$ for all $x \in X$. Hence, $q_K|_X = p_K$. Notice that q_K is a positively homogeneous, subadditive, and weak* lower semicontinuous function on X^{**}. We have $w^*\text{-cl}_{X^{**}}(B) \subset A$. Suppose that there is $x_0^{**} \in A$ such that $x_0^{**} \notin w^*\text{-cl}_{X^{**}}(B)$. By the separation theorem, there is $x_0^* \in X^*$ and $r \in R$ such that

$$\langle x_0^{**}, x_0^* \rangle > r > \sup\{\langle x_0^*, x\rangle;\ x \in B\} \geq 0.$$

It follows that $\sup\{\langle \frac{1}{r} x_0^*, x\rangle,\ x \in B\} \leq 1$. By the separation theorem, it is easy to verify $\frac{1}{r} x_0^* \in w^*\text{-cl}(\mathrm{co}(K))$. Hence $q_K(x_0^{**}) \geq \langle x_0^{**}, \frac{1}{r} x_0^* \rangle > 1$. This contradicts $x_0^{**} \in A$. ∎

EXAMPLE 2.1. Let K be a nonempty bounded subset of X^*, and let $f(x) = \sup\{\langle x^*, x\rangle;\ x^* \in K\}$ for all $x \in X$. Then

$$\hat{f}(x^{**}) = \sup\{\langle x^{**}, x^*\rangle;\ x^* \in w^*\text{-cl}(\mathrm{co}(K))\}, \qquad x^{**} \in X^{**}.$$

In particular, when $f(x) = \|x\|$ for all $x \in X$, $\hat{f}(x^{**}) = \|x^{**}\|$ for all $x^{**} \in X^{**}$.

Proof. Let $g(x^{**}) = \sup\{\langle x^{**}, x^*\rangle;\ x^* \in w^*\text{-cl}(\mathrm{co}(K))\}$ for all $x^{**} \in X^{**}$; then $g|_X = f$. By the weak* lower semicontinuity of g, $w^*\text{-cl}_{X^{**}}(X) = X^{**}$ and (II), $g(x^{**}) \leq \hat{f}(x^{**})$ for all $x^{**} \in X^{**}$. Suppose that there is $x_0^{**} \in X^{**}$ such that $g(x_0^{**}) < \hat{f}(x_0^{**})$. Pick a $x_0^* \in K$ and $r > 0$ such that

$$g(x_0^{**}) - \langle x_0^{**}, x_0^* \rangle < r < \hat{f}(x_0^{**}) - \langle x_0^{**}, x_0^* \rangle.$$

Let $A = \{x^{**} \in X^{**};\ g(x^{**}) - \langle x^{**}, x_0^* \rangle \leq r\}$ and $B = \{x \in X;\ f(x) - \langle x_0^*, x \rangle \leq r\}$. By Lemma 2.1, $A = w^*\text{-cl}_{X^{**}}(B)$. Hence $x_0^{**} \in w^*\text{-cl}_{X^{**}}(B)$, and so there is a net $\{x_\alpha\}$ in B such that $x_\alpha \xrightarrow{w^*} x_0^{**}$. It follows that

$$\hat{f}(x_0^{**}) \leq \liminf_\alpha f(x_\alpha) \leq \langle x_0^{**}, x_0^* \rangle + r.$$

Hence $\hat{f}(x_0^{**}) - \langle x_0^{**}, x_0^* \rangle \leq r$, a contradiction. ∎

3. VERY DIFFERENTIABLE CONVEX FUNCTIONS

It is an aspect of Banach spaces theory to study very smoothness of a space (see [11] and references therein). A point x in X with $\|x\| = 1$ is said to be a very differentiable point of X if there is a unique x^{***} in X^{***} such that $\|x^{***}\| = 1$ and $\langle x^{***}, x\rangle = 1$. This is equivalent to the norm of X^{**} being Gateaux differentiable at x. This and Example 2.1 incline us to introduce the very differentiability of convex functions.

DEFINITION 3.1. Let f be a continuous convex function defined on an open convex subset D of X. We say that f is very differentiable at $x \in D$ if $\hat{f}$, as a function on X^{**}, is Gateaux differentiable at x.

THEOREM 3.1. *Let f be a continuous convex function defined on an open convex subset D of X, and $x_0 \in D$. Then for each $x^{***} \in \partial\hat{f}(x_0)$ there is a net $\{x_\alpha\}$ in D and a net $\{x^*_\alpha\}$ in X^* such that $x_\alpha \to x_0$, $x^*_\alpha \xrightarrow{w^*} x^{***}$ and for each index α, $x^*_\alpha \in \partial f(x_\alpha)$.*

Proof. Let $C(f) = \{(x^*, r) \in X^* \times R;\ \langle x^*, x\rangle + rt \le 1,\ (x, t) \in \operatorname{epi}(f) - (x_0, f(x_0) + 1)\}$. We claim that for any $x^{***} \in \partial\hat{f}(x_0)$, $(x^{***}, -1) \in w^*\text{-cl}_{X^{***}\times R}(C(f))$. Indeed, suppose that there is $x_0^{***} \in \partial\hat{f}(x_0)$ such that $(x_0^{***}, -1) \notin w^*\text{-cl}_{X^{***}\times R}(C(f))$. By the separation theorem, there is $(x_0^{**}, r_0) \in X^{**} \times R$ and $t_0 \in R$ such that

$$\langle x_0^{***}, x_0^{**}\rangle - r_0 > t_0 > \sup\{\langle x_0^{**}, x^*\rangle + r_0 r;\ (x^*, r) \in C(f)\}.$$

Since $(0, 0) \in C(f)$, $t_0 > 0$. Hence

$$\left\langle x_0^{***}, \frac{1}{t_0} x_0^{**}\right\rangle - \frac{r_0}{t_0} > 1 \tag{1}$$

and

$$\sup\left\{\left\langle \frac{1}{t_0} x_0^{**}, x^*\right\rangle + \frac{r_0 r}{t_0};\ (x^*, r) \in C(f)\right\} < 1.$$

This and the separation theorem imply

$$\left(\frac{1}{t_0} x_0^{**}, \frac{r_0}{t_0}\right) \in w^*\text{-cl}_{X^{**}\times R}\big(\operatorname{epi}(f) - (x_0, f(x_0) + 1)\big).$$

By Proposition 2.1, $((1/t_0)x_0^{**}, (r_0/t_0)) \in \operatorname{epi}(\hat{f}) - (x_0, f(x_0) + 1)$. Hence there is $s_0 \geq 0$ such that

$$\frac{r_0}{t_0} = \hat{f}\left(\frac{1}{t_0}x_0^{**} + x_0\right) + s_0 - f(x_0) - 1.$$

This and (1) imply

$$\begin{aligned}\left\langle x_0^{***}, \frac{1}{t_0}x_0^{**}\right\rangle &> \hat{f}\left(\frac{1}{t_0}x_0^{**} + x_0\right) + s_0 - f(x_0) \\ &\geq \hat{f}\left(\frac{1}{t_0}x_0^{**} + x_0\right) - \hat{f}(x_0).\end{aligned}$$

This contradicts $x_0^{***} \in \partial\hat{f}(x_0)$. Hence for any $x^{***} \in \partial\hat{f}(x_0)$, there is a net $\{(x_\alpha^*, r_\alpha)\}$ in $C(f)$ such that $(x_\alpha^*, r_\alpha) \overset{w^*}{\to} (x^{***}, -1)$ in $X^{***} \times R$, that is, $x_\alpha^* \overset{w^*}{\to} x^{***}$ in X^{***} and $r_\alpha \to -1$. By $(x_\alpha^*, r_\alpha) \in C(f)$, for each $x \in D$,

$$\langle x_\alpha^*, x - x_0\rangle + r_\alpha\big(f(x) - f(x_0) - 1\big) \leq 1,$$

that is,

$$\langle x_\alpha^*, x - x_0\rangle \leq f(x) - f(x_0) + (1 + r_\alpha)\big(f(x_0) - f(x) + 1\big). \quad (2)$$

By the continuity of f, there is $\delta > 0$ such that $B_X(x_0, \delta) \subset D$ and $|f(x) - f(x_0)| < 1$ for all $x \in B_X(x_0, \delta)$. Let $\varepsilon_\alpha = 2|1 + r_\alpha|$; then $\varepsilon_\alpha \to 0$. Setting $g = f|_{B_X(x_0, \delta)}$, by (2), $\langle x_\alpha^*, x - x_0\rangle \leq g(x) - g(x_0) + \varepsilon_\alpha$ for each $x \in B_X(x_0, \delta)$. Hence $x_\alpha^* \in \partial_{\varepsilon_\alpha} g(x_0)$. By the Brondsted−Rockafellar theorem (see [10, p. 51, Theorem 3.18], there are $x_\alpha \in B_X(x_0, \delta)$ and $y_\alpha^* \in \partial g(x_\alpha) = \partial f(x_\alpha)$ such that

$$\|x_\alpha - x_0\| \leq \sqrt{\varepsilon_\alpha} \to 0 \quad \text{and} \quad \|y_\alpha^* - x_\alpha^*\| \leq \sqrt{\varepsilon_\alpha} \to 0.$$

By $x_\alpha \overset{w^*}{\to} x^{***}$, we have $y_\alpha^* \overset{w^*}{\to} x^{***}$. ∎

Remark. Clearly, $\partial f(x) \subset \partial\hat{f}(x)$ for all $x \in D$. This and Theorem 3.1 imply that for all $x \in D$,

$$\partial\hat{f}(x) = w^* - \lim_{y \to x} \partial f(y).$$

PROPOSITION 3.1. *Let f be a continuous convex function defined on an open convex subset D of X. Then $\partial\hat{f}(x) = w^* - \mathrm{cl}_{X^{***}}(\partial f(x))$ if and only if for each weak neighborhood W of 0 in X^* there is $\delta > 0$ such $\partial f(B_X(x, \delta)) \subset \partial f(x) + W$.*

Proof. The "if" part, it is easy to verify $w^*\text{-cl}_{X^{***}}(\partial f(x)) \subset \partial \hat{f}(x)$. Suppose that there is $x_0^{***} \in \partial \hat{f}(x)$ such that $x_0^{***} \notin w^*\text{-cl}_{X^{***}}(\partial f(x))$. By the separation theorem, there are $x_0^{**} \in X^{**}$ and $r \in R$ such that

$$\langle x_0^{***}, x_0^{**} \rangle > r > \sup\{\langle x_0^{**}, x^* \rangle;\ x^* \in \partial f(x)\}.$$

Let $W = \{x^* \in X^*;\ \langle x_0^{**}, x^* \rangle < k\}$, where $k = r - \sup\{\langle x_0^{**}, x^* \rangle;\ x^* \in \partial f(x)\}$. Then W is a weak neighborhood of 0 in X^*. Hence there is $\delta > 0$ such that $B_X(x, \delta) \subset D$ and $\partial f(B_X(x, \delta)) \subset \partial f(x) + W$. By Theorem 3.1, there is a net $\{x_\alpha\}$ in D and a net $\{x_\alpha^*\}$ in X^* such that

$$x_\alpha^* \in \partial f(x_\alpha), \quad x_\alpha \to x \quad \text{and} \quad x_\alpha^* \xrightarrow{w^*} x_0^{***}.$$

Without loss of generality we can assume that $\{x_\alpha\} \subset B_X(x, \delta)$. Hence for each index α, $x_\alpha^* \in \partial f(x) + W$, and so $\langle x_0^{**}, x_\alpha^* \rangle < r$. This and $x_\alpha^* \xrightarrow{w^*} x_0^{***}$ imply $\langle x_0^{***}, x_0^{**} \rangle \le r$, a contradiction. Hence $\partial \hat{f}(x) = w^*\text{-cl}_{X^{***}}(\partial f(x))$. For the "only if" part, it suffices to show that for any weak neighborhood W of 0 of the form

$$W = \left\{x^* \in X^*; |\langle x_i^{**}, x^* \rangle| < \varepsilon,\ x_i^{**} \in X^{**},\ i = 1, \dots, n \text{ and } \varepsilon > 0\right\},$$

there is $\delta > 0$ such that $\partial f(B_X(x, \delta)) \subset \partial f(x) + W$. Since $\partial \hat{f}$ is $\|\cdot\|$-w^* upper semicontinuous at $x \in D \subset \text{int}_{X^{**}}(\text{dom}(\hat{f}))$, there is $\delta > 0$ such that $\partial \hat{f}(B_{X^{**}}(x, \delta)) \subset \partial \hat{f}(x) + \frac{1}{2}\hat{W}$, where

$$\hat{W} = \left\{x^{***} \in X^{***};\ |\langle x^{***}, x_i^{**} \rangle| < \varepsilon,\ i = 1, \dots, n\right\}.$$

Notice that $\partial \hat{f}(x) = w^*\text{-cl}_{X^{***}}(\partial f(x)) \subset \partial f(x) + \frac{1}{2}\hat{W}$, $\frac{1}{2}\hat{W} + \frac{1}{2}\hat{W} \subset \hat{W}$ and $\partial f(B_X(x, \delta)) \subset \partial \hat{f}(B_X(x, \delta))$. One has $\partial f(B_X(x, \delta)) \subset \partial f(x) + W$. ∎

It is worth comparing Proposition 3.1 with Theorem 2.1 in [13].

COROLLARY 3.1. *Let f be a continuous convex function defined on an open convex subset D of X, and $x \in D$. Then $\partial \hat{f}(x) = \partial f(x)$ if and only if $\partial f(x)$ is a weak compact subset of X^* and ∂f is $\|\cdot\|$-w upper semicontinuous at x.*

Proof. The "only if" part is trivial. It suffices to show that "if" part. By the weak compactness of $\partial f(x)$, we have $w^*\text{-cl}_{X^{***}}(\partial f(x)) = \partial f(x)$. This and Proposition 3.1 imply $\partial \hat{f}(x) = \partial f(x)$. ∎

The following theorem is an immediate result of Corollary 3.1.

THEOREM 3.2. *Let f be a continuous convex function defined on an open convex subset D of X, and $x \in D$. Then f is very differentiable at x if and only if $\partial f(x)$ is a singleton and ∂f is $\|\cdot\|$-w upper semicontinuous at x.*

Theorem 3.2 explains that, for a continuous convex function f, the very differentiability of f is between the Gateaux differentiability of f and the Frechet differentiability of f. Moreover, when X is a nonreflexive Banach space, in general, the very differentiability of f is not any kind of the β-differentiability of f.

LEMMA 3.1. *Let K be a weak* compact convex subset of X^*, and ∂p_K be $\|\cdot\|$-w upper semicontinuous at $x_0 \in D$. Assume that a net $\{x_\alpha^*\}_{\alpha \in \Lambda}$ in K satisfies $\langle x_\alpha^*, x_0 \rangle \to p_K(x_0)$. Then there is a subnet $\{y_\beta^*\}_{\beta \in \Delta}$ of $\{x_\alpha^*\}$ and a net $\{z_\beta^*\}_{\beta \in \Delta}$ in $\partial_{pK}(x_0)$ such that $y_\beta^* - z_\beta^* \xrightarrow{w} 0$.*

Proof. Let $\varepsilon_\alpha = p_K(x_0) - \langle x_\alpha^*, x_0 \rangle$; then $\varepsilon_\alpha \geq 0$ and $\varepsilon_\alpha \to 0$. For each $h \in X$,

$$\begin{aligned}\langle x_\alpha^*, h \rangle = \langle x_\alpha^*, x_0 + h \rangle - \langle x_\alpha^*, x_0 \rangle &\leq p_K(x_0 + h) - \langle x_\alpha^*, x_0 \rangle \\ &= p_K(x_0 + h) - p_K(x_0) + \varepsilon_\alpha .\end{aligned}$$

Hence $x_\alpha^* \in \partial_{\varepsilon_\alpha} p_k(x_0)$. By the Brondsted–Rockafellar theorem, there is a net $\{x_\alpha\}_{\alpha \in \Lambda}$ in X and a net $\{a_\alpha^*\}_{\alpha \in \Lambda}$ in X^* such that

$$a_\alpha^* \in \partial p_K(x_\alpha),\ \|x_\alpha - x_0\| \leq \sqrt{\varepsilon_\alpha} \to 0 \text{ and } \|a_\alpha^* - x_\alpha^*\| \leq \sqrt{\varepsilon_\alpha} \to 0. \quad (3)$$

Let $\Delta = \{(\alpha, U);\ \alpha \in \Lambda$ and U is a weak neighborhood of 0 in $X^*\}$, and define a partial order on Δ as follows: for $(\alpha_1, U_1), (\alpha_2, U_2) \in \Delta$, $(\alpha_1, U_1) \leq (\alpha_2, U_2)$ if and only if $\alpha_1 \leq \alpha_2$ and $U_2 \subset U_1$. It is clear that $(\Delta, \leq)$ is a directed set. Since ∂p_K is $\|\cdot\|$-w upper semicontinuous at x_0, by (3), for each $(\alpha, U) \in \Delta$, there is $\varphi(\alpha, U) \in \Lambda$ such that $\alpha \leq \varphi(\alpha, U)$ and $a_{\varphi(\alpha,U)}^* \in \partial p_K(x_0) + U$. Hence there is $b_{\varphi(\alpha,U)}^* \in \partial p_K(x_0)$ such that $a_{\varphi(\alpha,U)}^* - b_{\varphi(\alpha,U)}^* \in U$. Denote $x_{\varphi(\alpha,U)}^*$ and $b_{\varphi(\alpha,U)}^*$ by $y_{(\alpha,U)}^*$ and $z_{(\alpha,U)}^*$, respectively. It is easy to verify that $\{y_{(\alpha,U)}^*\}_{(\alpha,U) \in \Delta}$ is a subnet of $\{x_\alpha^*\}_{\alpha \in \Lambda}$, $\{z_{(\alpha,U)}^*\}_{(\alpha,U) \in \Delta}$ is a net in $\partial p_K(x_0)$, and $y_{(\alpha,U)}^* - z_{(\alpha,U)}^* \xrightarrow{w} 0$. ∎

PROPOSITION 3.2. *Let K be a weak* compact convex subset of X^*, and $x_0 \in X$. The following properties are equivalent*:

(i) *Each point x^* in $\partial p_K(x_0)$ is a weak*-weak continuous point of K ($i.e.$, each net in K is weakly convergent to x^* whenever it is weak* convergent to x^*).*

(ii) *Each net $\{x_\alpha^*\}$ in K with $\langle x_\alpha^*, x_0 \rangle \to p_K(x_0)$ has a subnet which is weakly convergent to some point in $\partial p_K(x_0)$.*

(iii) $\partial \hat{p}_K(x_0) = \partial p_K(x_0)$.

Proof. (i) $\Rightarrow$ (ii). By weak* compactness of K, each net $\{x_\alpha^*\}$ in K with $\langle x_\alpha^*, x_0 \rangle \to p_K(x_0)$ has a subnet which is weak* convergent to some x^* in K. It follows that $\langle x^*, x_0 \rangle = p_K(x_0)$. By [10, Lemma 5.10], $x^* \in \partial p_K(x_0)$. This and (i) imply that $\{x_\alpha^*\}$ has a subnet which is weakly convergent to x^*.

(ii) $\Rightarrow$ (iii). It is clear from (ii) that $\partial p_K(x_0)$ is weakly compact. Suppose that $\partial \hat{p}_K(x_0) \neq \partial p_K(x_0)$. By Corollary 3.1, there is a weak open set W including $\partial p_K(x_0)$, a sequence $\{x_n\}$ in X and a sequence $\{x_n^*\}$ in X^* such that $x_n \to x_0$, $x_n^* \in \partial p_K(x_n)$ and $x_n^* \notin W$ for all n. Since

$$\langle x_n^*, x_0 \rangle = \langle x_n^*, x_n \rangle + \langle x_n^*, x_0 - x_n \rangle$$
$$= p_K(x_n) + \langle x_n^*, x_0 - x_n \rangle \to p_K(x_0),$$

without loss of generality we can assume that there is $x^* \in K$ such that $x_n^* \xrightarrow{w} x^*$. It follows that $x^* \notin W$ and $\langle x^*, x_0 \rangle = p_K(x_0)$. This and Lemma 5.10 in [10] implies $x^* \in \partial p_K(x_0)$. This contradicts $\partial p_K(x_0) \subset W$.

(iii) $\Rightarrow$ (i). Let x_0^* be any point in $\partial p_K(x_0)$ and $\{x_\beta^*\}$ be any net in K such that $x_\beta^* \xrightarrow{w^*} x_0^*$. For any subnet $\{y_\alpha^*\}$ of $\{x_\beta^*\}$, by Lemma 3.1 and the weak compactness of $\partial p_K(x_0)$, $\{y_\alpha^*\}$ has a subnet which is weakly convergent. This implies $x_\beta^* \xrightarrow{w} x_0^*$. ∎

Let K be a weak* compact subset of X^* and K_0 be the weak* closed convex hull of K. Notice that for each $x \in X$ there is $x^* \in K$ such that

$$\langle x^*, x \rangle = \sup\{\langle y^*, x \rangle;\ y^* \in K\} = \sup\{\langle y^*, x \rangle;\ y^* \in K_0\}.$$

One has the following corollaries.

COROLLARY 3.2. *Let K be a weakly compact subset of X^*. Then $\partial \hat{p}_K(x) = \partial p_K(x)$ for all $x \in X$.*

COROLLARY 3.3. *Let K be a weak* compact convex subset of X^* and $x_0 \in X$. Then p_K is very differentiable at x_0 if and only if each net $\{x_\alpha^*\}$ in K with $\langle x_\alpha^*, x_0 \rangle \to p_K(x_0)$ is weakly convergent.*

COROLLARY 3.4. *Let K be a weak* compact subset of X^* and $x_0 \in X$. If $\partial \hat{p}_K(x_0) = \partial p_K(x_0)$, there are $x_1 \in X$ and $x_0^* \in K$ such that each net $\{x_\alpha^*\}$ in K with $\langle x_\alpha^*, x_i \rangle \to \langle x_0^*, x_i \rangle$ $(i = 0, 1)$ is weakly convergent to x_0^*. In particular, x_0^* is a weak*-weak continuous point of K.*

Proof. Let K_0 be the weak* closed convex hull of K. By Corollary 3.1, $\partial p_K(x_0)$ is a weakly compact convex subset of K_0. Hence there are $x_1 \in X$ and $x_0^* \in \partial p_K(x_0)$ such that x_1 strongly exposes $\partial p_K(x_0)$ at x_0^*. Notice that $\partial p_K(x_0)$ is an extremal subset of K_0 and x_0^* is an extremal point of $\partial p_K(x_0)$. x_0^* is an extremal point of K_0. Hence $x_0^* \in K$ (see [2, Theorem 36.10]). Let $\{x_\alpha^*\}$ be any net in K with $\langle x_\alpha^*, x_i \rangle \to \langle x_0^*, x_i \rangle$ $(i = 0, 1)$. By Proposition 3.2, it is sufficient to prove that each subnet of $\{x_\alpha^*\}$ is weakly convergent to x_0^* whenever the subnet is weakly convergent. Assume that a subnet $\{y_\beta^*\}$ of $\{x_\alpha^*\}$ is weakly convergent. By Lemma 3.1, there is a subnet $\{a_\gamma^*\}$ of $\{y_\beta^*\}$ and a net $\{b_\gamma^*\}$ in $\partial p_K(x_0)$ such that $a_\gamma^* - b_\gamma^* \xrightarrow{w} 0$. This implies $\langle b_\gamma^*, x_1 \rangle \to \langle x_0^*, x_1 \rangle$. Hence $\|b_\gamma^* - x_0^*\| \to 0$. It follows that $\{y_\beta^*\}$ is weakly convergent to x_0^*. ∎

4. A CHARACTERIZATION OF ASPLUND SPACES

In this section we will give an characterization of Asplund spaces which is similar to Theorem 2.7 in [4]. We need the following lemma.

LEMMA 4.1 ([4, Theorem 1.3]). *Let Φ be a minimal w^*-cusco from a complete metric space A into the bidual space X^{**} of a Banach space X. Assume that the set*

$$E = \{x \in A;\ \Phi(x) \subset w^*\text{-cl}_{X^{**}}(\Phi(x) \cap X)\}$$

is a residual subset of A (i.e., E includes a dense G_δ subset of A). Then Φ is single-valued and upper semicontinuous with respect to the norm topology of X^{**} at each point in a dense G_δ subset of A.

PROPOSITION 4.1. *X is an Asplund space if and only if for each bounded subset K of X^* there is a metric ρ on K such that every bounded subset of K admits weak* slices of arbitrarily small ρ-diameter and the topology induced by ρ on K is stronger than or equal to the weak topology.*

Proof. The "only if" part is immediate from Theorem 2.32 in [10]. To prove the "if" part, it suffices to show that, for each open convex subset D of X and each continuous convex function f defined on D, f is Frechet differentiable on a dense subset of D (because the set of all points of D where f are Frechet differentiable is a G_δ subset of D). Since the subdifferential mapping ∂f is locally bounded, for each $x \in D$ and each neighborhood U of x there is an open convex neighborhood V of x such that $V \subset U \cap D$ and $\partial f(V)$ is bounded. Hence there is a metric ρ on $\partial f(V)$ such that every bounded subset of $\partial f(V)$ admits weak* slices of arbitrarily small ρ-diameter and the topology induced by ρ on $\partial f(V)$ is stronger than or equal to the weak topology. For each natural number n, let $G_n = \{x \in V$; there is an open subset W of V such that $x \in W$ and $\rho - \operatorname{diam}(\partial f(W)) < \frac{1}{n}\}$. Similar to the proof of [10, Theorem 2.30], we have that $G = \bigcap_{n=1}^\infty G_n$ is a dense G_δ subset of V. It is clear that for each $x \in G$, $\partial f(x)$ is a singleton and ∂f is $\|\cdot\|$-ρ upper semicontinuous at x, and so ∂f is $\|\cdot\|$-w upper semicontinuous at x. It is sufficient to show that f is Frechet differentiable on a dense subset of V. Let T be the restriction of ∂f to G. Then T is a $\|\cdot\|$-w continuous single-valued mapping from G into X^*. Hence T is a minimal w-cusco from G into X^*. Since a dense G_δ subset of a complete metric space is itself completely metrizable ([5, p.

96]), by Lemma 4.1 there is a dense G_δ subset A of G such that for each $x \in A$, T is $\|\cdot\|$-$\|\cdot\|$ upper semicontinuous at x. This and Lemma 2.5 in [4] imply that ∂f, as a set-valued mapping from V into X^*, is single-valued and $\|\cdot\|$-$\|\cdot\|$ upper semicontinuous at each point of A. Hence f is Frechet differentiable at each point of the dense subset A of V. ∎

In view of Proposition 4.1 the following problem is natural.

Is X an Asplund space if every bounded subset of X^* admits arbitrarily small weak* slices with respect to the weak topology (i.e., for any weak neighborhood W of 0 in X^* and any bounded subset K of X^* there are $x \in X$ and $\alpha > 0$ such that $S(x, K, \alpha) - S(x, K, \alpha) \subset W$, where $S(x, K, \alpha) = \{x^* \in K;\ \langle x^*, x\rangle > \sup\{\langle y^*, x\rangle;\ y^* \in K\} - \alpha\}$)?

Consider the James Tree space JT which is a separable Banach space with nonseparable dual space. In addition every infinite dimensional subspace of JT contains a copy of l_2 (see [3]). Hence JT is not an Asplund space and contains no copy of l_1. Notice that X contains no copy of l_1 if and only if every bounded subset K of X^* admits arbitrarily small weak* slices with respect to the weak topology (see [14, pp. 351–352]). It follows that JT is not an Asplund space and every bounded subset of JT^* admits arbitrarily small weak* slices with respect to the weak topology.

G. Godefroy [7] really gave the following characterization of an Asplund space:

X is an Asplund space if and only if each equivalent norm on X has a very differentiable point in X (see the proof of Theorem II.2 and Remark II.3 in [7]).

It follows directly from Godefroy's result and Corollary 3.3 that X is an Asplund space if and only if for each weak* compact convex subset K of X^* there are $x \in X$ and $x^* \in K$ such that each net $\{x^*_\alpha\}$ is weakly convergent to x^* whenever $\langle x^*_\alpha, x\rangle \to \langle x^*, x\rangle = p_K(x)$. But the existence of weak*-weak continuous points of each weak* compact convex subset of X^* does not imply that X is an Asplund space. Indeed, Ghoussoub, Maurey, and Schachermayer [6] have shown that the existence of weak*-norm continuous points of each weak* compact convex subset of X^* does not imply that X is an Asplund space.

The following result is an equivalent form of Theorem 3.6 in Moors [8].

X is an Asplund space if and only if, for each weak* compact subset K of X^*, the set of weak*−weak continuous points of K is residual in (K, w^*).

In view of above-presented results and Corollary 3.4, we present the following problem:

Is X an Asplund space if, for each equivalent norm $|||\cdot|||$ (each continuous seminorm p) on X, there is $x \in X$ such that $\partial\,|||\hat{\cdot}|||(x) = \partial\,|||\cdot|||(x)(\partial\hat{p}(x) = \partial p(x))$?

Note added in Proof. Giles [15] answered the above problem "yes" sometime after this paper was submitted.

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